Lecture-4

Gauss's Law – Maxwell's equation

Gauss' Law

Mathematically*, we express the idea two slides back as

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{q_{enclosed}}{\varepsilon_o}$$

Gauss' Law

We will find that Gauss law gives a simple way to calculate electric fields for charge distributions that exhibit a high degree of symmetry…

…and save more complex and realistic charge distributions for advanced classes.

To see how this works, let's do an example.

Example: use Gauss' Law to calculate the electric field from an isolated point charge q.

To apply Gauss' Law, we construct a "Gaussian Surface" enclosing the charge.

The Gaussian surface should mimic the symmetry of the charge distribution.

For this example, choose for our Gaussian surface a sphere of radius r, with the point charge at the center.

I'll work the rest of the example on the blackboard.

Strategy for Solving Gauss' Law Problems

- Select a Gaussian surface with symmetry that matches the charge distribution.

- Draw the Gaussian surface so that the electric field is either constant or zero at all points on the Gaussian surface.

- Use symmetry to determine the direction of $\vec{E}$ on the Gaussian surface.

- Evaluate the surface integral (electric flux).

- Determine the charge inside the Gaussian surface.

- Solve for $\vec{E}$.

Example: use Gauss' Law to calculate the electric field due to a long line of charge, with linear charge density λ.

Example: use Gauss' Law to calculate the electric field due to an infinite sheet of charge, with surface charge density σ.

These are easy using Gauss' Law (remember what a pain they were in the previous chapter). Study these examples and others in your text!

$$E_{line} = \frac{\lambda}{2\pi\varepsilon_0 r}.$$

$$E_{sheet} = \frac{\sigma}{2\varepsilon_0}.$$

The top 5 reasons why we make you learn Gauss' Law:

5. You can solve (high-symmetry) problems with it.

4. It's good for you. It's fun! What more can you ask!

3. It's easy. Smart physicists go for the easy solutions.

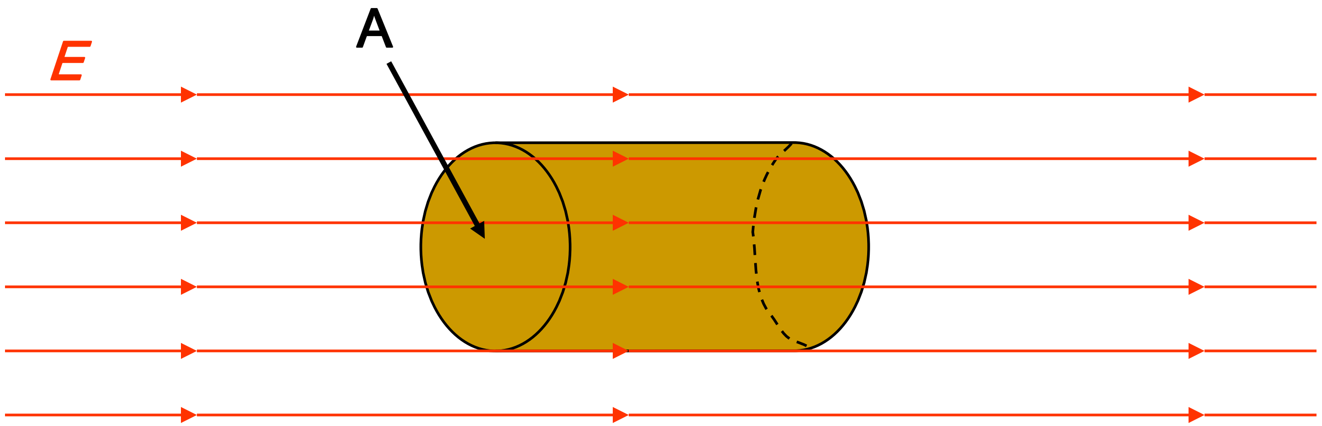
2. If I had to learn it, you do too.

And the number one reason…

…will take a couple of slides to present

Worked Example 1

Compute the electric flux through a cylinder with an axis parallel to the electric field direction.



The flux through the curved surface is zero since **E** is perpendicular to *d***A** there. For the ends, the surfaces are perpendicular to **E**, and **E** and **A** are parallel. Thus the flux through the left end (*into* the cylinder) is –*EA*, while the flux through right end (*out* of the cylinder) is +*EA*. Hence the net flux through the cylinder is zero.

Gauss's Law

Gauss's Law relates the electric flux through a *closed* surface with the charge Q_{in} inside that surface.

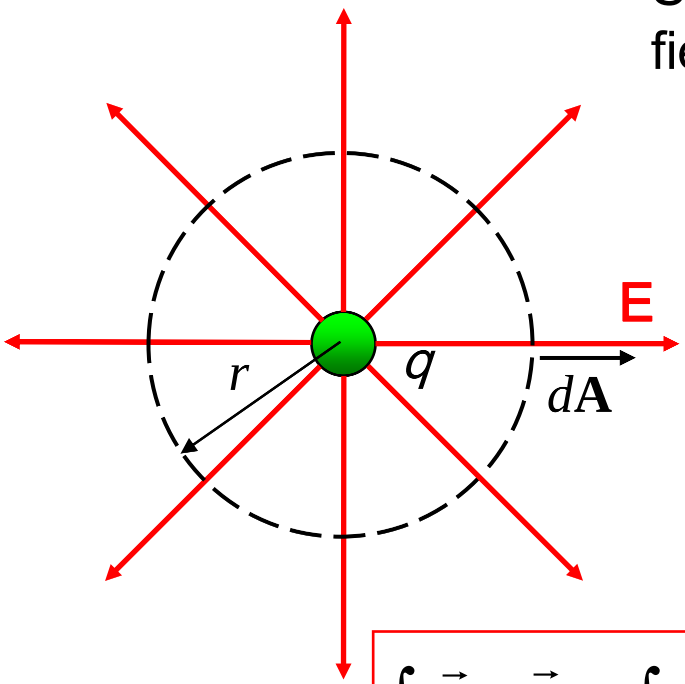
$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{Q_{in}}{\varepsilon_0}$$

This is a useful tool for simply determining the electric field, but only for certain situations where the charge distribution is either rather simple or possesses a high degree of symmetry.

Problem Solving Strategies for Gauss's Law

- Select a Gaussian surface with symmetry that matches the charge distribution
- Draw the Gaussian surface so that the electric field is either constant or zero at all points on the Gaussian surface
- Use symmetry to determine the direction of **E** on the Gaussian surface
- Evaluate the surface integral (electric flux)
- Determine the charge inside the Gaussian surface
- Solve for **E**

Worked Example 2



Starting with Gauss's law, calculate the electric field due to an isolated point charge *q*.

We choose a Gaussian surface that is a sphere of radius *r* centered on the point charge. I have chosen the charge to be positive so the field is radial outward by symmetry and therefore everywhere perpendicular to the Gaussian surface.

$$\vec{E}\cdot d\vec{A} = E\,dA$$

Gauss's law then gives:

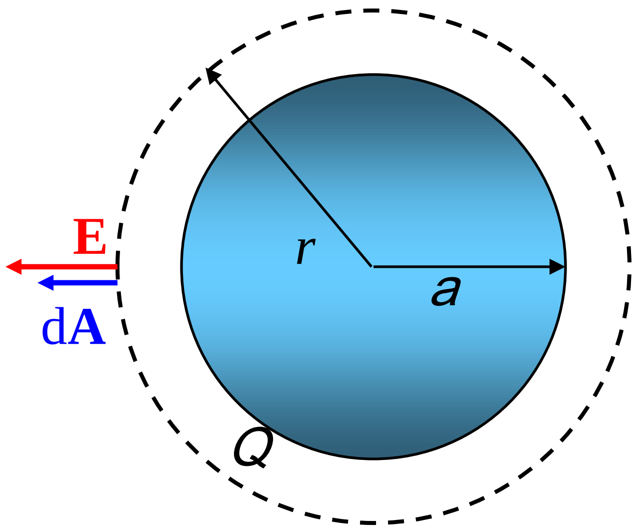
$$\oint \vec{E}\cdot d\vec{A} = \oint E\,dA = \frac{Q_{in}}{\varepsilon_0} = \frac{q}{\varepsilon_0}$$

Symmetry tells us that the field is constant on the Gaussian surface.

$$\oint E\,dA = E\oint dA = E\left(4\pi r^2\right) = \frac{q}{\varepsilon_0} \quad \text{so} \quad E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} = k_e\frac{q}{r^2}$$

Worked Example 3

An insulating sphere of radius *a* has a uniform charge density ρ and a total positive charge *Q*. Calculate the electric field outside the sphere.



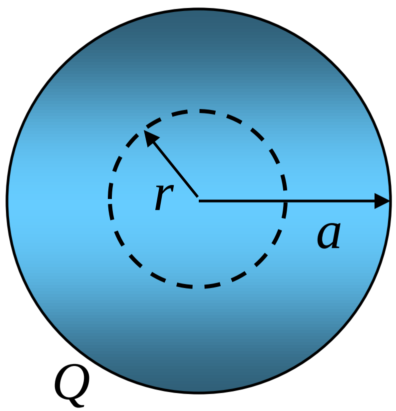
Since the charge distribution is spherically symmetric we select a spherical Gaussian surface of radius $r > a$ centered on the charged sphere. Since the charged sphere has a positive charge, the field will be directed radially outward. On the Gaussian sphere **E** is always parallel to d**A**, and is constant.

❶ Left side: $\oint \vec{E} \cdot d\vec{A} = \oint E\, dA = E \oint dA = E\left(4\pi r^2\right)$

❷ Right side: $\frac{Q_{in}}{\varepsilon_0} = \frac{Q}{\varepsilon_0}$

❸ $E\left(4\pi r^2\right) = \frac{Q}{\varepsilon_0}$ or $E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2} = k_e \frac{Q}{r^2}$

Worked Example 3 cont'd



Find the electric field at a point inside the sphere.

Now we select a spherical Gaussian surface with radius *r* < *a*. Again the symmetry of the charge distribution allows us to simply evaluate the left side of Gauss's law just as before.

❶ $$\text{Left side: } \oint \vec{E}\cdot d\vec{A} = \oint E\,dA = E\oint dA = E\left(4\pi r^2\right)$$

The charge inside the Gaussian sphere is no longer *Q*. If we call the Gaussian sphere volume *V'* then

❷ $$\text{Right side: } Q_{in} = \rho V' = \rho\frac{4}{3}\pi r^3$$

❸ $$E\left(4\pi r^2\right) = \frac{Q_{in}}{\varepsilon_0} = \frac{4\rho\pi r^3}{3\varepsilon_0}$$

❹ $$E = \frac{4\rho\pi r^3}{3\varepsilon_0\left(4\pi r^2\right)} = \frac{\rho}{3\varepsilon_0}r \quad \text{but} \quad \rho = \frac{Q}{\frac{4}{3}\pi a^3} \quad \text{so} \quad E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{a^3}r = k_e\frac{Q}{a^3}r$$

Worked Example 3 cont'd

We found for $r > a$, $E = k_e \frac{Q}{r^2}$

and for $r < a$, $E = \frac{k_e Q}{a^3} r$

Let's plot this: